## Lattices and Boolean Algebras

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Abstract $\qquad$ In this work, we study the main properties of lattices and Boolean Algebras ,we give various examples. In particular we study the finite Boolean Algebras . Indeed it is isomorphic to $P(\mathrm{X})$ for some x .

Finally , as an application we study the algebra of electrical circuits.

## Introduction

The axioms of a ring give structure to the operations of addition and multiplication on a set. However; we can construct algebraic structures, known as Lattices and Boolean Algebras that generalize other types of operations. For example, the important operations on sets are inclusion, union, and intersection. Lattices are generalization of order relation on algebraic spaces, such as set inclusion in set theory and inequality in the familiar number systems $\mathrm{N}, \mathrm{Z}, \mathrm{Q}$, and R. Boolean algebras have found application in logic, circuit theory, and probability.

## Lattices and Boolean Algebras Partially Order Sets

We begin by the study of lattices and Boolean algebras by generalizing the idea of inequality. Recall that a relation on a set $X$ is a subset of $X \times X$. A relation $P$ on $X$ is called a partial order of $X$ if it satisfies the following axioms:

1. The relation is reflexive : $(a, a) \in P$ for all $a \in X$.
2. The relation is ant symmetric: if $(a, b)$ $\in P$ and $(b, a) \in P$ then $a=b$.
3. The relation is transitive: if $(a, b) \in$ $P$ and $(b, c) \in P$ then $(a, c) \in P$.

We will usually write $a \preceq b$ to mean $(a, b) \in P$ unless some symbol is naturally associated with a particular partial order ,such as $\mathrm{a} \preceq \mathrm{b}$ with integers $a$ and $b$, or $X \subseteq Y$
with sets $X$ and $Y$. A set $X$ together with a partial order is called a partial order set, or poset .

## Example:

The set of integer (or rational or reals) is poset where $a \leq b$. has the usual meaning for two integers $a$ and $b$ in $\mathbf{Z}$.

## Example:

Let $X$ be any set. We will define the power set of $X$ to be the set of all subset of $X$. We denote the power set of $X$ by $P(X)$ for example, let $X=\{a, b, c\}$.Then $P(X)$ is the set of all subset of the set $\{a, b, c\}$ :

$$
\begin{array}{cccc}
\phi & \{a\} & \{b\} & \{c\} \\
\{a, b\} & \{b, c\} & \{a, c\} & \{a, b, c\}
\end{array}
$$

On any power set of a set, set inclusion $\subseteq$ is a partial order. We can represent the order on $\{a, b, c\}$ schematically by a diagram such as the on in Figure 1.1


Figure 1 partial order on $\mathrm{P}(\{a, b, c\})$.

## Example:

let $G$ be a group, the set of subgroup of $G$ is a poset where the partial order is set inclusion.
Proof:
Let $G$ is a group then:

1. $G \subseteq G$ since any group subset of it self so $G$ is reflexive.
2. if $G_{1} \subseteq G_{2} \& G_{2} \subseteq G_{1}$ Then $G_{1}=G_{2}$ So $G$ is ant symmetric
3.if $G_{1} \subseteq G_{2} \& G_{2} \subseteq G_{3}$,Then $G_{1} \subseteq G_{3}$,then $G$ is transitive.

## Example:

There can be more than one partial order on a particular set. We can form a partial order on $\mathbb{N}$ by $a \prec b$ if $a \mid b$.

## Proof:

The relation is certainly reflexive since $a \mid$ $a$ for all $a \in \mathbb{N}$.
If $m \mid n$ and $n \mid m$, then $m=n$; hence the relation is also antisymmetric.
The relation is transitive, because if $m \mid n$ and $n \mid p$,then $m \mid p$.

## Example:

let $X=\{1,2,3,4,6,8,12,24\}$ be a set of divisors of 24 with the partial order define in Example 4. Figure 1.2 shows the partial order on $X$.


Figure 2 partial order on the divisors of 24.

## Last upper bound and great lower bound Definition :

1- Let $Y$ be a subset of a poset $X$. An element u in $X$ is an upper bound of Y if $a$ $\prec u$ for every element $a \in Y$. If u is an upper bound of $Y$ such that
$u \preceq v$ for every other upper bound $\quad v$ of $Y$, then $u$ is called a least upper bound or supremum of $Y$.

2- An element $l$ in $X$ is said to be a lower bound of $Y$ if $l \prec a$ for all $a \in Y$. If $l$ is a lower bound of $Y$ such that $k \preceq l$ for every other lower bound $k$ of $Y$, then $l$ is called a greatest lower bound or infmum of $Y$

## Example:

Let $Y=\{2,3,4,6\}$ be contained in the set $X$ of Example 5.
Then $Y$ has upper bounds 12 and 24, with 12 as a least upper bound. The only lower bound is 1 ; hence, it must be a greatest lower bound.
As it turns out, least upper bounds and greatest lower bounds are unique if they exist.

## Theorem :

Let $Y$ be a nonempty subset of a poset $X$. If $Y$ has a least upper bound, then $Y$ has a unique least upper bound. If $Y$ has a greatest lower bound, then $Y$ has a unique greatest lower bound.

## Proof:

Let $u_{1}$ and $u_{2}$ be least upper bounds for . By the definition of the least upper bound, $u_{1} \preceq u$ for all upper bounds $u$ of $Y$. In particular, $u_{1} \preceq u_{2}$. Similarly, $u_{2} \preceq u_{1}$. Therefore, $u_{1}=u_{2}$ by antisymmetry. A similar argument show that the greatest lower bound is unique.

## Definition:

A lattice is a poset $L$ such that every pair of elements in $L$ has a least upper bound and a greatest lower bound. The least upper bound of $a, b \in L$ is called the join of $a$ and $b$ and is denoted by $a \vee b$. The greatest lower bound of $a, b \in L$ is called the meet of $a$ and $b$ and is denoted by $a \wedge b$.

## Example 7:

Let $X$ be a set. Then the power set of $X$, $P(X)$, is a lattice.
For two sets $A$ and $B$ in $P(X)$, the least upper bound of $A$ and $B$ is $A \cup B$.
Certainly $A \cup B$ is an upper bound of $A$ and $B$, since $A \subseteq A \cup B$ and
$B \subseteq A \cup B$. If $C$ is some other set containing both $A$ and $B$, then $C$ must contain $A \cup B$; hence, $A \cup B$ is the least upper bound of $A$ and $B$. Similarly, the greatest lower bound of $A$ and $B$ is
$A \cap B$.

## Definition:

Principle of Duality. Any statement that is true for all lattices remains
true when $\preceq$ is replaced by $\succeq$ and $\vee$ and $\wedge$ are interchanged throughout the statement.
The following theorem tells us that a lattice is an algebraic structure with two binary operations that satisfy certain axioms.

## Theorem:

If $L$ is a lattice, then the binary operations $\vee$ and $\wedge$ satisfy
the following properties for $a, b, c \in L$.

1. Commutative laws: $a \vee b=b \vee a$ and $a b=b \wedge a$.
2. Idempotent laws:
$a \vee a=a$ and $a \wedge a=a$.
3. Associative laws:
$a \vee(b \vee c)=(a \vee b) \vee c$ and
$a \wedge(b \wedge c)=(a \wedge b) \wedge c$.
4. Absorption laws:
$a \vee(a \wedge b)=a$ and
$a \wedge(a \vee b)=a$.

## Proof.

By the Principle of Duality, we need only prove the first statement in each part.

1. By definition $a \vee b$ is the least upper bound of $\{a, b\}$, and $\mathrm{b} \vee \mathrm{a}$ is the least
upper bound of $\{b, a\}$ however, $\{a, b\}=$ $\{b, a\}$.
2.The join of $a$ and $a$ is the least upper bound of $\{a\}$; hence, $a \vee a=a$.
2. We will show that $a \vee(b \vee c)$ and $(a \vee b) \vee c$ are both least upper bounds of $\{a, b, c\}$. Let $d=a \vee b$. Then $c \preceq$ $d \vee c=(a \vee b) \vee c$. We also know that

$$
\begin{aligned}
a \prec a \vee b & =d \prec d \vee c \\
& =(a \vee b) \vee c .
\end{aligned}
$$

A similar argument demonstrates that $b \preceq$ $(a \vee b) \vee c$. Therefore, $(a \vee b) \vee c$ is an upper bound of $\{a, b, c\}$. We now need to show that $(a \vee b) \vee c$ is the least upper bound of $\{a, b, c\}$. Let $u$ be some other upper bound of $\{a, b, c\}$.
Then $a \preceq u$ and $\preceq u$; hence,
$d=a \vee b$ 々 $u$.
Since $c \vee u$, it follows that
$(\mathrm{a} \vee \mathrm{b}) \vee \mathrm{c}=\mathrm{d} \vee \mathrm{c} \preceq \mathrm{u}$. Therefore,
$(a \vee b) \vee c$ must be the least upper bound of $\{a, b, c\}$. The argument that shows $a \vee(b \vee c)$ is the least upper bound of $\{a, b, c\}$ is the same. Consequently,

$$
a \vee(b \vee c)=(a \vee b) \vee c
$$

4. Let $d=a \wedge b$. Then $a \preceq a \vee d$. On the other hand, $d=a \wedge b \preceq a$, and so $a \vee d \preceq a$. Therefore, $a \vee(a \wedge b)=a$.

## Theorem:

Let $L$ be a nonempty set with two binary operations $\vee$ and $\wedge$ satisfying the commutative, associative, idempotent, and absorption laws.
We can define a partial order on $L$ by $a \preceq$ $b$ iff $a \vee b=b$. Furthermore, $L$ is a lattice with respect to $\preceq$ if for all $a, b \in$ $L$, we define the least upper bound and
greatest lower bound of $a$ and $b$ by $a \vee b$ and $a \wedge b$, respectively.

## Proof :

We first show that $L$ is a poset under $\preceq$.
Since $\vee a=a, a \prec a$ and $\prec$ is reflexive. To show that $\preceq$ is ant symmetric, let $a \preceq b$ and $b \preceq a$. Then $a \vee b=b$ and $b \vee a=a$. By the commutative law,
$b=a \vee b=b \vee a=a$.
Finally, we must show that $\prec$ is transitive. Let $a \preceq b$ and $b \preceq c$. Then $a \vee b=b$ and $b \vee c=c$. Thus,

$$
\begin{aligned}
a \vee c & =a \vee(b \vee c) \\
& =(a \vee b) \vee c \\
& =b \vee c=c,
\end{aligned}
$$

or $a \preceq c$
To show that $L$ is a lattice, we must prove that $a \vee b$ and $a \vee b$ are, respectively, the least upper and greatest lower bounds of $a$ and $b$.
Since
$a=(a \vee b) \wedge a=a \wedge(a \vee b)$,
it follows that $a \preceq a \vee b$.
Similarly, $b \preceq a \vee b$. Therefore,
$a \vee b$ is an upper bound for $a$ and $b$. Let $u$ be any other upper bound of both $a$ and $b$. Then $a \preceq u$ and $b \preceq u$. But $a \vee b$

$$
\begin{aligned}
\frac{\prec u \text { since }}{(a \vee b) \vee u} & =a \vee(b \vee u) \\
& =a \vee u \\
& =u .
\end{aligned}
$$

## Boolean Algebra

Let $u$ is investigate the example of the power set, $P(X)$, of a set $X$ more closely. The power set is a lattice that is ordered by inclusion. By the definition of the power set, the largest element in $P(X)$ is $X$ itself and the smallest element is $\phi$, the empty set. For any set $A$ in $P(X)$, we know that
$A \cap X A$ and $A \cup \phi=A$. This suggests the following definition for lattices. An element $I$ in a poset $X$ is a largest element if $a \preceq I$ for all
$a \in X$. An element $O$ is a smallest element of $X$ if $O \preceq a$ for all $a \in X$.

Let $A$ be in $P(X)$. Recall that the complement of $A$ is
$A^{\prime}=X \backslash A$
$=\{x: x \in X$ and $x \notin A\}$.
We know that $A \cup A^{\prime}=X$ and
$A \cap A^{\prime}=\phi$. We can generalize this example for lattices. A lattice $L$ with a largest element $I$ and a smallest element $O$ is complemented if for each $a \in X$, there exists an a' such that $a \vee a^{\prime}=I$ and $a \wedge a^{\prime}=0$.
In a lattice $L$, the binary operations $\vee$ and $\wedge$ satisfy commutative and associative laws; however, they need not satisfy the distributive law

$$
a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c)
$$

however, in $P(X)$ the distributive law is satisfied since
$A \cap(B \cup C)=(A \cap B) \cup(A \cap C)$
for $A, B, C \in P(X)$. We will say that a lattice $L$ is distributive if the
following distributive law holds:
$a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c)$
for all $a, b, c \in L$.

## Theorem:

A lattice $L$ is distributive if and only if $a \vee(b \wedge c)=(a \vee b) \wedge(a \vee c)$ for all $a, b, c \in L$.

## Proof:

Let us assume that is a distributive lattice.

$$
\begin{aligned}
& a \vee(b \wedge c) \\
& =[a \vee(a \wedge c)] \vee(b \wedge c) \\
& =a \vee[(a \wedge c) \vee(b \wedge c)] \\
& =a \vee[(c \wedge a) \vee(c \wedge b)] \\
& =a \vee[c \wedge(a \vee b)]
\end{aligned}
$$

$=a \vee[(a \vee b) \wedge c]$
$=[(a \vee b) \wedge a] \vee[(a \vee b) \wedge c]$
$=(a \vee b) \wedge(a \vee c)$.
The converse follows directly from the Duality Principle.

## Definition:

A Boolean algebra is a lattice $B$ with a greatest element $I$ and a smallest element $O$ such that $B$ is both distributive and complemented.

The power set of $X, P(X)$, is our prototype for a Boolean algebra. As it turns out, it is also one of the most important Boolean algebras. The following theorem allows us to characterize Boolean algebras in terms of the binary relations $\wedge$ and $\vee$ without mention of the fact that a Boolean algebra is a poset.

## Theorem:

$A$ set $B$ is a Boolean algebra if and only if there exist binary operations $\vee$ and $\wedge$ on $B$ satisfying the following axioms.

1. $a \vee b=b \vee a$ and
$a \wedge b=b \wedge a$ for $a . b \in B$.
2. $a \vee(b \vee c)=(a \vee b) \vee c$ and
$a \wedge(b \wedge c)=(a \wedge b) \wedge c$
for $a, b, c \in B$.
3. $a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c)$
and
$a \vee(b \wedge c)=(a \vee b) \wedge(a \vee c)$
for $a, b, c \in B$.
4. There exist elements $I$ and $O$ such that $a \vee O=a$ and $a \wedge I=a$ for all $a \in B$.
5. For every $a \in B$ there exists an $a^{\prime} \in B$ such that $a \vee a^{\prime}=I$ and $a \wedge a^{\prime}=0$.

## Proof:

Let $B$ be a set satisfying (1) - (5) in the theorem. One of the idempotent laws is satisfied since

$$
\begin{aligned}
a & =a \vee o \\
& =a\left(a \wedge a^{\prime}\right. \\
& =(a \vee a) \wedge\left(a \vee a^{\prime}\right) \\
& =(a \vee a) \wedge I=a \vee a .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& I \vee b=(I \vee b) \wedge I= \\
& (I \wedge I) \vee(b \wedge I)=I \vee I=I
\end{aligned}
$$

Consequently, the first of the two absorption laws holds, since

$$
\begin{aligned}
a \vee(a \wedge b) & =(a \wedge I) \vee(a \wedge b) \\
= & a \wedge(I \vee \\
& =a \wedge I \quad=a
\end{aligned}
$$

The other idempotent and absorption laws are proven similarly. Since $B$ also satisfies (1) - (3), the conditions of Theorem are met, therefore, $B$ must be a lattice. Condition (4) tells us that $B$ is a distributive lattice.
For $a \in B, O \vee a=a$; hence, $O \prec a$ and $O$ is the smallest element in $B$.
To show that I is the largest element in $B$, we will first show that $a \vee b=b$ is equivalent to $a \wedge b=a$. Since $a \vee I=a$ for all $a \in B$, using the absorption laws we can determine that

$$
\begin{aligned}
a \vee I & =(a \wedge I) \vee I \\
& =I \vee(I \wedge a)=I
\end{aligned}
$$

or $a \prec I$ for all a in $B$. Finally, since we know that $B$ is complemented by (5), $B$ must be a Boolean algebra.

Conversely, suppose that $B$ is a Boolean algebra. Let $I$ and $O$ be the
greatest and least elements in $B$, respectively. If we define $a \vee b$ and $a \wedge b$ as least upper and greatest lower bounds of $\{a, b\}$, then $B$ is $a$ Boolean algebra by Theorem, our hypothesis.

Many other identities hold in Boolean algebras. Some of these identities are listed in the following theorem.

## Theorem:

Let $B$ be a Boolean algebra. Then

1. $a \vee I=I$ and $a \wedge O=O$
for all $a \in B$.
2. If $a \vee b=a \vee c$ and
$a \wedge b=a \wedge c$ for $a, b, c \in B$, then $b=c$.
3. If $a \vee b=I$ and $a \wedge b=O$, then $b=a^{\prime}$.
4. $\left(a^{\prime}\right)^{\prime}=a$ for all $a \in B$.
5. $I^{\prime}=O$ and $O^{\prime}=I$.
6. $(a \vee b)^{\prime}=a^{\prime} \wedge b^{\prime}$ and $(a \wedge b)^{\prime}=$ $a^{\prime} \vee b^{\prime}$ (De Morgan's Laws).

## Proof:

1) $a \vee I=(a \wedge I) \vee I$

$$
=I \vee(I \wedge a)=I
$$

2) $a \vee b=a \vee c$ and $a \wedge b=$ $a \wedge c$, we have

$$
\begin{aligned}
b & =b \vee(b \wedge a) \\
& =b \vee(a \wedge b) \\
& =b \vee(a \wedge c) \\
& =(b \vee a) \wedge(b \vee c) \\
& =(a \vee b) \wedge(b \vee c) \\
& =(a \vee c) \wedge(b \vee c) \\
& =(c \vee a) \wedge(c \vee b)
\end{aligned}
$$

$$
\begin{aligned}
& =c \vee(a \wedge b) \\
& =c \vee(a \wedge c) \\
& =c \vee(c \wedge a) \\
& =c .
\end{aligned}
$$

$$
\begin{aligned}
& \text { 3) }(a \vee b)=\left(a \vee a^{\prime}\right) \text { and } \\
& (a \wedge b)=\left(a \wedge a^{\prime}\right) \longrightarrow \mathrm{By}(2) \longrightarrow \\
& b=a^{\prime} .
\end{aligned}
$$

4) $\operatorname{By}(3)\left(a \wedge a^{\prime}\right)=O$ and $\left(a \vee a^{\prime}\right)=I\left(\mathrm{a}^{\prime}\right)^{\prime}=\mathrm{a}$.
5) $I=a \vee a^{\prime}$, then
$\left(I=a \vee a^{\prime}\right)$

$$
\begin{aligned}
I^{\prime} & =a^{\prime} \wedge\left(a^{\prime}\right)^{\prime} \\
& =a^{\prime} \wedge a \\
& =0
\end{aligned}
$$

And $O=a \wedge a^{\prime}$, then

$$
\left(O=a \wedge a^{\prime}\right)^{\prime}
$$

$$
O^{\prime}=a^{\prime} \vee\left(a^{\prime}\right)^{\prime}
$$

$$
O^{\prime}=a^{\prime} \vee a
$$

$$
=I
$$

6) $(a \vee b)^{\prime}=a^{\prime} \wedge b^{\prime}$, since
$(a \vee b) \vee\left(a^{\prime} \vee b^{\prime}\right)$
$=\left(a \vee b \vee a^{\prime}\right) \wedge\left(a \vee b \vee b^{\prime}\right)$
$=\left(\left(a \vee a^{\prime}\right) \vee b\right) \wedge\left(a \vee\left(b \vee b^{\prime}\right)\right)$
$=(I \vee b) \wedge(a \vee I)$
$=I$
Similarly $(a \vee b) \wedge\left(a^{\prime} \vee b^{\prime}\right)=0$

## Finite Boolean Algebras

A Boolean algebra is a finite Boolean algebra if it contains a finite number of elements as a set. Finite Boolean algebras are particularly nice since we can classify them up to isomorphism.
Let $B$ and $C$ be Boolean algebras. A bijective map $\Phi: B \rightarrow C$ is an isomorphism of Boolean algebras if

$$
\begin{aligned}
& \Phi(a \vee b)=\Phi(a) \vee \Phi(b) \\
& \Phi(a \wedge b)=\Phi(a) \wedge \Phi(b)
\end{aligned}
$$

for all $a$ and $b$ in $B$.

We will show that any finite Boolean algebra is isomorphic to the Boolean algebra obtained by taking the power set of some finite set $X$. We will need a few lemmas and definitions before we prove this result.

## Definition :

Let $B$ be a finite Boolean algebra. An element $a \in B$ is an atom of $B$ if $a \neq$ $O$ and $a \wedge b=a$ for all $b \in B$. Equivalently, $a$ is an atom of $B$ if there is no nonzero $b \in B$ distinct from $a$ such that $O$ 〔 $b$ 亿 $a$.

## Lemma:

Let $B$ be a finite Boolean algebra. If $b$ is a nonzero element of $B$, then there is an atom $a$ in $B$ such that $a \preceq b$.

## Proof:

If $b$ is an atom, let $a=b$. Otherwise, choose an element $b_{1}$, not equal to $O$ or $b$, such that $b_{l} \preceq b$. We are guaranteed that this is possible since $b$ is not an atom. If $b_{1}$ is an atom, then we are done. If not, choose $b_{2}$, not equal to $O$ or $b_{1}$, such that $b_{2} \preceq b_{1}$. Again, if $b_{2}$ is an atom, let $a=b_{2}$ continuing this process, we can obtain a chain

$$
0 \preceq \ldots \text { œ.. } b_{3} \preceq b_{2} \preceq b_{1} \preceq b .
$$

Since $B$ is a finite Boolean algebra, this chain must be finite. That is, for some $k$, $b_{k}$ is an atom. Let $a=b_{k}$.

## Lemma :

Let $a$ and $b$ be atoms in a finite Boolean algebra $B$ such that $a \neq b$.
Then $a \wedge b=0$.

## Proof:

Since $a \wedge b$ is the greatest lower bound of $a$ and $b$, we know that $a \wedge b \preceq a$.
Hence, either $a \wedge b=a$ or $a \wedge b=$ 0 . However, if $a \wedge b=a$, then either $a$ $\preceq b$ or $a=O$. In either case we have a
contradiction because $a$ and $b$ are both atoms; therefore, $a \wedge b=0$.

## Lemma:

Let $B$ be a Boolean algebra and
$a, b \in B$. The following statements are equivalent.

1. $a \prec b$.
2. $a \wedge b^{\prime}=0$.
3. $a^{\prime} \vee b=I$.

## Proof:

(1) $\longrightarrow$ (2). If $a \preceq b$, then $a \vee b=$ $b$. Therefore,

$$
\begin{aligned}
& a \wedge b^{\prime}=a \wedge(a \vee b)^{\prime} \\
&=a \wedge\left(a^{\prime} \wedge b^{\prime}\right) \\
&=\left(a \wedge a^{\prime}\right) \wedge b^{\prime} \\
&=0 \wedge b^{\prime} \\
&=0 . \\
& \begin{aligned}
&(2) \longrightarrow(3) . \text { If } a \wedge b^{\prime}=O, \text { then } \\
& a^{\prime} \vee b=\left(a \wedge b^{\prime}\right)^{\prime}=o^{\prime}=I .
\end{aligned} \\
& \text { (3) } \longrightarrow(1) . \text { If } a^{\prime} \vee b=I, \text { then } \\
& a=a \wedge\left(a^{\prime} \vee b\right) \\
&=\left(a \wedge a^{\prime}\right) \vee(a \wedge b) \\
&= O \vee(a \wedge b) \\
&= a \wedge b .
\end{aligned}
$$

## Lemma:

Let $B$ be $a$ Boolean algebra and $b$ and $c$ be elements in $B$ such that $b \nless c$.
Then there exists an atom $a \in B$ such that $a \preceq b$ and $a \npreceq c$.

## Proof:

By Lemma 2.3, $b \wedge c^{\prime} \neq O$. Hence, there exists an atom $a$ such that
$a \prec b \wedge c^{\prime}$. Consequently, $a \preceq b$ and $a \nsucceq c$.

## Lemma:

Let $b \in B$ and $a_{l}, \ldots, a_{n}$ be the atoms of $B$ such that $a_{i} \prec b$.
Then $b=a_{l} \vee \ldots \vee a_{n}$. Furthermore, if $a, a_{1}, \ldots, a_{n}$ are atoms of $B$ such that $a \prec$ $b, a_{i} \preceq b$, and
$b=a \vee a_{1} \vee \ldots \vee a_{\mathrm{n}}$, then $a=a_{\mathrm{i}}$ for some $i=1, \ldots, n$.

## Proof:

Let $b_{I}=a_{1} \vee \ldots \vee a_{n}$. Since $a_{i} \prec b$ for each $i$, we know that $b_{1} \preceq b$. If we can show that $b \nprec b_{1}$, then the lemma is true by ant symmetry. Assume $b \prec b_{1}$. Then there exists an atom $a$ such that $a \prec b$ and $a \prec \not b_{1}$. Since $a$ is an atom and $a \prec$ $b$, we can deduce that $a=a_{i}$ for some $a_{\mathrm{i}}$. However, this is impossible since $a \prec b_{l}$. Therefore, $b$ 〔 $b_{1}$.

Now suppose that $b=a_{1} \vee \ldots \vee a_{n}$. If $a$ is an atom less than $b$,

$$
\begin{aligned}
a & =a \wedge b \\
& =a \wedge\left(a_{l} \vee \ldots \vee a_{n}\right) \\
& =\left(a \wedge a_{l}\right) \vee \ldots \vee\left(a \wedge a_{n}\right) .
\end{aligned}
$$

But each term is $O$ or a with $a \wedge a_{i}$ occurring for only one $a_{\mathrm{i}}$. Hence, by
Lemma $a=a_{i}$ for some $i$.

## Theorem:

Let $B$ be a finite Boolean algebra.
Then there exists a set $X$ such that $B$ is isomorphic to $P(X)$.

## Proof:

We will show that $B$ is isomorphic to $P(X)$, where $X$ is the set of atoms of $B$.

Let $a \in B$. By Lemma, we can write a uniquely a $a=a_{1} \vee \ldots \vee a_{n}$ for $a_{1}, \ldots, a_{n} \in X$. Consequently, we can define a map

$$
\begin{aligned}
\Phi: B & \rightarrow P(X) \text { by } \\
\Phi(a) & =\Phi\left(a_{1} \vee \ldots \vee a_{n}\right) \\
& =\left\{a_{1}, \ldots, a_{n}\right\} .
\end{aligned}
$$

Clearly , $\Phi$ is onto.
Now let $a=a_{1} \vee \ldots \vee a_{n}$ and $b=b_{1}$ $\vee \ldots \vee b_{m}$ be elements in $B$, where each $a_{i}$ and each $b_{i}$ is an atom.

If $\Phi(a)=\Phi(b)$, then $\left\{a_{1}, \ldots, a_{n}\right\}=$ $\left\{b_{1}, \ldots, b_{m}\right\}$ and $a=b$. Consequently, $\Phi$ is injective.
The join of $a$ and $b$ is preserved by $\Phi$ since

$$
\begin{aligned}
& \begin{aligned}
& \Phi(a \vee b)=\Phi\left(a_{l} \vee \ldots \vee a_{n} \vee b_{1}\right. \\
&\left.\vee \ldots \vee b_{m}\right) \\
&=\left\{a_{l}, \ldots, a_{n}, b_{l}, \ldots, b_{m}\right\} \\
&=\left\{a_{l}, \ldots, a_{n}\right\} \cup\left\{b_{l}, \ldots, b_{m}\right\} \\
&= \Phi\left(a_{l} \vee \ldots \vee a_{n}\right) \cup \\
& \Phi\left(b_{l} \vee \ldots \vee b_{m}\right. \\
&= \Phi(a) \cup \Phi(b) .
\end{aligned} \\
& \text { Similarly ; } \Phi(a \wedge b)=\Phi(a) \cap \Phi(b) .
\end{aligned}
$$

## corollary:

The order of any finite Boolean algebras must be $2^{\mathrm{n}}$ for some positive integer n .

## The Algebra of Electrical Circuits

The usefulness of Boolean algebras has become increasingly apparent over the past several decades with the development
of the modern computer . The circuit design of computer chips can be expressed in terms of Boolean algebras. In this section we will develop the Boolean algebra of electrical circuits and switches; however, these results can easily be generalized to the design of integrated computer circuitry.

A switch is a device, located at some point in an electrical circuit, that controls the flow of current through the circuit. Each switch has two possible states: it can be open, and not allow the passage of current through the circuit, or a it can be closed, and allow the passage of current. These states are mutually exclusive. We require that every switch be in one state or the other: a switch cannot be open and closed at the same time. Also, if one switch is always in the same state as another, we will denote both by the same letter that is, two switches that are both labeled with the same letter $a$ will always be open at the same time and closed at the same time.

Given two switches, we can construct two fundamental types of circuits. Two switches $a$ and $b$ are in series if they make up a circuit of the type that is illustrated in Figure 3.3. Current can pass between the terminals $A$ and $B$ in a series circuit only if both of the switches a and b are closed. We will denote this combination of switches by $a \wedge b$. Two switches $a$ and $b$ are in parallel if they form a circuit of the type that appears in Figure 3.4. In the case of a parallel circuit, current can pass between $A$ and $B$ if either one of the switches is closed. We denote a parallel combination of circuits $a$ and $b$ by $a \vee b$.
$A-a-b-B$

Figure 3.3. $a \wedge b$


We can build more complicated electrical circuits out of series and parallel circuits by replacing any switch in the circuit with one of these two fundamental types of circuits. Circuits constructed in this manner are called series-parallel circuits.

We will consider two circuits equivalent if they act the same. That is, if we set the switches in equivalent circuits exactly the same we will obtain the same result. For example, in a series circuit $a \wedge b$ is exactly the same as $b \wedge a$. Notice that this is exactly the commutative law for Boolean algebras. In fact, the set of all series-parallel circuits forms a Boolean algebra under the operations of $\checkmark$ and $\wedge$. We can use diagrams to verify the different axioms of a Boolean algebra. The distributive law,
$a \wedge(b \vee c=(a \wedge b) \vee(a \wedge c)$
is illustrated in Figure 3.5. If $a$ is a switch, then $a^{\prime}$ is the switch that is always open when $a$ is closed and always closed when $a$ is open. A circuit that is always closed is $I$ in our algebra; a circuit that is always open is $O$. The laws for $a \wedge a^{\prime}=O$ and $\mathrm{a} \vee \mathrm{a}^{\prime}=\mathrm{I}$ are shown in Figure 3.6.


Figure 3.5.

$$
\begin{gathered}
a \wedge(b \vee c)= \\
(a \wedge b) \vee(a \wedge c)
\end{gathered}
$$

## Example:

Every Boolean expression represents a switching circuit. For example, given the expression
$(a \vee b) \wedge\left(a \vee b^{\prime}\right) \wedge(a \vee b)$, we can construct the circuit in Figure 3.7.

## Theorem:

The set of all circuits is a Boolean algebra.
We can now apply the techniques of Boolean.

$$
-a-a^{\prime}-
$$



Figure 3.6.
$a \wedge a^{\prime}=O$ and $a \vee a^{\prime}=I$

Figure
3.7. $(a \vee b) \wedge\left(a \vee b^{\prime}\right) \wedge(a \vee b)$
algebras to switching theory.

## Example:

Given a complex circuit, we can now apply the techniques of Boolean algebra to reduce it to a simpler one. Consider the circuit in Figure 3.7. Since

$$
\begin{aligned}
&(a \vee b) \wedge\left(a \vee b^{\prime}\right) \wedge(a \vee b) \\
&=(a \vee b) \wedge(a \vee b) \wedge\left(a \vee b^{\prime}\right) \\
&=(a \vee b) \wedge\left(a \vee b^{\prime}\right) \\
&=a \vee\left(b \vee b^{\prime}\right) \\
&=a
\end{aligned}
$$

we can replace the more complicated circuit with a circuit containing the single switch a and achieve the same function.


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